Deployment of a Radial Spacecraft Formation

Using Direct Multiple Shooting with Nonlinear Programming

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The deployment of a radial spacecraft formation is sought using nonlinear programming with direct multiple-shooting. Such a formation can be used to simultaneously collect data at multiple points in space. The optimal control of multiple spacecraft is rephrased as a finite-parameter, constrained, optimization problem that is then solved by nonlinear programming. The cases that are solved include the initialization and deployment of a three-spacecraft, radially-aligned formation. The initial conditions for a periodic formation are determined. These conditions are then used as the final positions and velocities in the optimization of the spacecraft deployment.

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I. Introduction

Formation flight in space has been the subject of much research meant to address maneuvers ranging from the rendezvous of two spacecraft\(^1\) to the station-keeping of multiple spacecraft in a virtual aperture\(^2\). Determining the optimal initialization and deployment of formations depend on such things as the type of orbit desired, the model of the dynamics, and the control input that is available. The perspective taken in this paper is to find such optimal maneuvers by a numerical optimization approach. The benefit of this approach is its ability to accommodate scenarios that cannot be solved analytically. This paper will first discuss formulating a generic optimal control problem. It is only possible to solve a small fraction of optimal control problems analytically; for example, an orbital transfer problem under point-mass gravity models with some thruster assumptions can be handled analytically\(^3\). The following section will describe how to numerically solve optimization problems using direct multiple shooting with nonlinear programming to transform the time-continuous optimal control problem into a finite-parameter, constrained optimization problems. The last section will discuss how the technique can be applied to spacecraft formations. It will first present how the system is modeled and how its performance is assessed. This is followed by an application of the technique to spacecraft formation initialization and deployment.

The approach taken here has been used before to determine optimal trajectories of individual spacecraft: converting the optimal control problem into a parameter optimization problem that can be solved by nonlinear programming. Hull\(^4\) provides an excellent synopsis of the various approaches used in converting optimal control problems into parameter optimization problems. The approaches are categorized into four classes based on the unknown parameters that will be varied and the numerical integration schemes that will be used.
In Class 1, the control parameters are unknown, and the state differential equations are integrated explicitly; this is also referred to as direct shooting. Hull and Speyer\textsuperscript{5} use this approach in solving the optimal plane-change of individual spacecraft. In Class 2, the control parameters and some or all of the state parameters are unknown, and the state differential equations are integrated explicitly; this is referred to as direct multiple-shooting and is the approach that will be used in this paper. In Class 3, the control parameters and all of the state parameters are unknown, and the state differential equations are usually integrated implicitly and referred to as direct collocation. Hargraves and Paris\textsuperscript{6}, Enright and Conway\textsuperscript{7}, and Betts\textsuperscript{8,9} extensively discuss this technique and its application to individual spacecraft. Milam et al.\textsuperscript{10} discuss this approach as applied to satellite formation maintenance and reconfiguration. In Class 4, the state parameters are unknown, and the state differential equations are integrated implicitly; when the control parameters are then analytically determined, the technique is called differential inclusion. Seywald\textsuperscript{11} illustrates this approach applied to the one-dimensional rocket-ascent problem.

Dealing with multiple spacecraft in a formation has been addressed in multiple ways. Formation flight in deep-space can often be modeled by ignoring all celestial bodies thus resulting in linear equations of motion among the spacecraft in the formation\textsuperscript{12,13}. Formation flight in the vicinity of one of the libration points between two celestial bodies is often modeled by Hill's linear equations for the restricted, three-body problem\textsuperscript{14,15,16} which also result in linear equations of motion. Formation flight in low-orbit around a celestial body must contend with the nonlinear dynamics which arise from basic gravitational attraction, oblateness of the main attracting body, and atmospheric drag. A common approach in modeling the dynamics of the relative motion of the formation is to assume that the spacecraft are traveling closely to a reference orbit. When the reference orbit is circular and the only component of that dynamics
model is an inverse-square gravity field, the relative equations of motion are described by the linear, Clohessy-Wiltshire (CW) equations\textsuperscript{17}. Progress has been made on extending these relative equations of motion to also account for 1\textsuperscript{st}-order nonlinear terms\textsuperscript{18}, 2\textsuperscript{nd}-order nonlinear terms\textsuperscript{19}, J2 gravitational effects\textsuperscript{20,21}, higher-order gravitational effects\textsuperscript{22}, and atmospheric drag\textsuperscript{23,24}. When the reference orbit is elliptical and the dynamics are modeled by an inverse-square gravity field then the relative equations of motion are described by the linear, Tschauner-Hempel equations\textsuperscript{25,26,27}.

One area of interest in the design of spacecraft formations is finding the periodic motion of the spacecraft within a formation such that little to no control effort need be applied to maintain the formation. In the literature, setting up such types of formations is referred to as initialization. Koon et al.\textsuperscript{28} have found such behavior for near-circular orbits under the influence of J2 gravitational effects. Inhalan et al.\textsuperscript{29} and Vaddi et al.\textsuperscript{30} found periodic formations for eccentric orbits.

Another area of interest is finding the optimal control necessary to maintain or establish a formation when uncontrolled, periodic motion is either unattainable or simply not desired. When the relative equations of motion are linear, it is often convenient to control the various spacecraft by the use of a linear-quadratic-regulator\textsuperscript{19}. This has been applied to determine optimal trajectories for a single pair of satellites\textsuperscript{32,33} and for larger formations\textsuperscript{34,35,36}. However, Inalhan et al.\textsuperscript{37} emphasize the limitations on the use of linearized, relative dynamics: "For close proximity formations on the order of a few hundred meters, the linearized dynamics provide useful and precise models for formation flight design. However, for larger or longer maneuvers, which can take more than a few orbits, the effects of measurement noise, nonlinear orbital effects, and differential disturbances will cause deviations in the final relative states." Furthermore, there are times when analytic solutions for the optimal control of formation spacecraft is either
inconvenient or intractable. The reasons stem from the nonlinearity of the mathematical models of the dynamics and/or constraints on individual spacecraft and/or constraints on the coupled motion of multiple spacecraft. Yeh et al.\textsuperscript{38} approach this problem by using a nonlinear controller designed around the assumed linear motion with simulations showing how it can accommodate the actual nonlinear, relative motion. Strizzi et al.\textsuperscript{39} discuss the use of Legendre pseudospectral methods to numerically solve generic optimal control problems.

In this paper, we use direct multiple shooting with nonlinear programming to investigate the initialization and deployment of a formation. The first scenario involves optimizing the propellant-consumption for a three-spacecraft, periodic, radial formation by varying each spacecraft’s initial conditions. These initial conditions are then used as the final conditions for several scenarios involving the deployment of this formation. The approach used in this paper finds solutions that take advantage of the nonlinear, orbital dynamics to minimize propellant consumption.

**II. Nonlinear Optimal Control Problems and their Solutions**

In this paper, the state of the system will be described by a state vector $\mathbf{x}(t)$ which is a function of time, the control law will be represented by $\mathbf{k}(t)$, and the nonlinear system dynamics will be described by the function $\mathbf{f}$. Hence

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, \mathbf{u}, t).$$

(1)

An assessment of the optimality of the control law will be quantified by a scalar cost function, $J$:

$$J(\mathbf{x}, \mathbf{k}, t) = \int_{t}^{t_f} \left[ l(\mathbf{x}, \mathbf{k}, \tau) + m(\mathbf{x}(\tau)) \right] d \tau.$$

(2)

This generic cost function integrates the instantaneous cost $l$ for the duration of the time
interval of interest, $t$ to $t_f$, and appends a terminal penalty, $m$, that accounts for any deviance from the final desired state. The cost is a scalar value but the state of the system, controller, equality, and inequality constraints may be vectors of different sizes.

### A. Solving Optimal Control Problems with Nonlinear Programming

The technique we use attempts to make the problem tractable by approximating a time-continuous optimal control problem as a finite-parameter, constrained optimization problem. This problem is further transformed into a finite-parameter, unconstrained optimization problem that can be solved by nonlinear programming. This last transformation is taken because we found that it decreases the frequency of getting caught in poor local minima given a zero-control initial guess. The following is a brief description of the approach.

1) *Discretizing and Formulating the Parameter Optimization Problem*

The first step in discretizing the optimal control problem is to divide the entire time interval into a finite number of subintervals, with the times at each end of a subinterval being called nodes. The time-continuous control law is then discretized at these nodes, and assembled as a set of variables at each node. The variables from each node are then aggregated into a vector of variables (design variables). If necessary to the optimal control problem being solved, additional variables (such as some free initial conditions) may also be appended to the vector of design variables.

\[
Z = \begin{bmatrix}
\hat{F}_1 \\
\hat{F}_2 \\
\vdots \\
\hat{F}_i \\
\vdots \\
\hat{F}_n \\
\hat{r}_{\text{init}} \\
\hat{v}_{\text{final}}
\end{bmatrix}
\]  

(3)
Eq. (3) shows the design variable vectors for some example problems. $\vec{F}_i$ is the parameterized control vector at node $i$, an instant in time, and $\vec{r}_{\text{initial}}$ is the free initial position and $\vec{v}_{\text{initial}}$ is the free initial velocity for the optimal control of interest.

The result of the discretization transforms the time-continuous constraint on the system dynamics, as described by Eq. (1), into a finite number of equality constraints. The equality constraints arising in this work consist of the residual error between the desired final positions and velocities versus the propagated final positions and velocities. The residual final error is computed for each of the spacecraft.

$$h_{\text{vehicle}} = \begin{bmatrix} \vec{r}_{\text{desired}} - \vec{r}_{\text{propagated}} \\ \vec{v}_{\text{desired}} - \vec{v}_{\text{propagated}} \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{0} \end{bmatrix}$$

(4)

Similarly, all relevant functions, such as the cost function, must be redefined to accommodate the design variables vector rather than a time-continuous control law, $k(t)$. Consider the original cost function, $J$:

$$J(x, k, t) = \int_{t}^{t_f} l(x, k, \tau) d\tau + M(x(t_f))$$

(5a)

After sub-dividing the controller histories into $(n-1)$ segments, the new cost function is defined as:

$$J(x, k, t) = \sum_{i=1}^{n-1} l(x_i, k_i, \tau_i) \Delta\tau + M(x(t_f))$$

(5b)

Aside from the nonlinear system dynamics, any other constraints may also be accommodated. Such constraints could include conditions on the initial and final states of the system, resulting in additional equality constraints. Bounds on the values of some design variables result in additional inequality constraints. Finally, the entire set of constraints are categorized into a set of equality constraints, $h(Z) = 0$, and inequality constraints, $g(Z) \leq 0$. 
The original optimal control problem is now represented by a vector of design variables with an accompanying cost function, equality constraint functions (including the system dynamics errors), and inequality constraint functions.

Minimize  \[ J(Z) = \sum_{i=1}^{n} l(z_i) \Delta \tau + M(z_n) \]  \hspace{1cm} (6)

subject to  \[ h(Z) = 0 \quad \text{and} \quad g(Z) \leq 0 \]

This constrained, nonlinear optimization problem is further converted into an unconstrained optimization problem by the use of an exterior penalty function (EPF). This consists of creating a new cost function composed of the original cost function plus a weighted sum of the constraint violations/residual errors in the equality and inequality constraints. The sum is weighted by a penalty parameter, \( r \). As the penalty parameter is gradually increased, the unconstrained optimization problem approaches the solution of the constrained problem.

Minimize  \[ J_{EPF}(Z) = J(Z) + \sum r \left[ |h(Z)|^2 + |\min(g(Z), 0)|^2 \right] \]  \hspace{1cm} (7)

2) Solving the Parameter Optimization Problem

We minimize this new unconstrained optimization problem by using the Davidson-Fletcher-Powell algorithm:

1. A guess is provided to the solver as a starting solution, in the form of a design variables vector.

2. Given the current solution, the solver determines the 1st derivative vector and an approximation to the 2nd derivative Hessian matrix of the exterior penalty function cost.

3. The algorithm determines a descent direction that reduces the exterior penalty function cost using the 1st and 2nd derivatives determined in the prior step.
4. A step length along the descent direction is calculated that reduces the the exterior penalty function cost as much as possible.

5. The solution is shifted down along that descent direction to acquire a new, improved solution.

6. Return to Step 2 and repeat until the exterior penalty function cost stops decreasing. (Ideally, the gradient vector of the exterior penalty function cost should be the zero vector.)

Once a minimum for the exterior penalty function cost, J_{EPF}, at a specified penalty parameter, r, has been found it must be compared with the original cost, J. If J_{EPF} is significantly greater than J, then the solution to the unconstrained optimization problem has not yet converged to a solution of the constrained problem. Therefore, if the solution has not yet converged, the optimization must be repeated with a greater penalty parameter, r, and with the latest solution used as the new guess for Step 1.

The sequence of solutions have converged to the final solution once the difference between the J_{EPF} and J has dropped below a user-specified tolerance value. With regards to the original constrained optimization problem, this difference is composed entirely of the violations in the equality and inequality constraints. In terms of trajectory optimization, this slack permits extra energy from the constraint violations to reduce the required control effort necessary to satisfy all constraints (especially greater velocities than specified by boundary constraints yielding higher kinetic energies resulting in lower control effort). Therefore, it is essential to minimize this slack so that the solution to the unconstrained problem properly reflects the true solution of the underlying constrained problem.

One further check run on all solutions of the unconstrained optimization problem is to check whether the solutions satisfy a necessary condition of optimality for the constrained optimization problem, known as the Karush-Kuhn-Tucker conditions. All of the solutions found in this work
satisfy these conditions to within 0.5% of the zero vector as measured relative to the objective function gradient value.

3) Approximating the Discretized Trajectory

The accuracy of the trajectory over the entire time interval of the problem is measured by a finite number of equality constraints as represented by Eq. (4). The state of the system at each node is propagated forward in time until the next node. This is repeated at each node, hence explaining the nomenclature of multiple-shooting\textsuperscript{41,42}. We specifically use the Runge-Kutta-Fehlberg numerical integration algorithm\textsuperscript{43} which is an explicit numerical integration scheme. The control applied at the start of each segment is maintained constant over the course of that segment. The propagated system state at the final time is then compared with the terminal boundary condition, and any difference between the two is a residual error.

Figure 1 is a graphical depiction of how multiple-shooting with nonlinear programming is used to find an optimal trajectory. The variables at each node, consisting of the piecewise-constant control input, and any free initial conditions are varied by the nonlinear programming solver to ensure that the system dynamics constraints are satisfied (in particular, that the violation/error is less than a desired tolerance.)
Fig. 1   Depiction of a nonlinear optimization solver finding an optimal trajectory.

The zoomed portion shows the trajectory of a spacecraft as it travels during the time interval between $t_i$ and $t_f$. At time $t_f$ the position and velocity of the spacecraft is compared against the final boundary position and velocity (the target). The difference between the two is the error, and is indicated by the red line in both the zoomed portion, and in the full state versus time plots.

The lower portion of the diagram displays three columns of plots. The top plot, state
versus time, displays a simplified, one-dimensional representation of the state of the spacecraft. The control versus time plot shows the control effort expended during each of the time intervals. The integral of cost versus time plot shows the total cost as it accumulates over time. (Note that the cost could be computed by any means desired including fuel consumption, obstacle clearance, etc.)

Each column represents one iteration of trajectory. It is the goal of the nonlinear optimization solver to both minimize the integral of the cost and to drive all of the errors to zero.

4) Smoothness of Functions

The assumption inherent in most nonlinear programming algorithms is that the functions and constraints that are supplied to it are smooth to first-order. Our use of the exterior penalty function (Eq. (7)) as a cost function works well to smooth out the inequality constraints which have discontinuous first-derivatives at the point where the inequality becomes violated. However, the exterior penalty function does not accommodate a cost function that is originally not-smooth. This issue arises when dealing with the propellant consumption, as described by Eq. (9b), which makes use of the absolute function; it has a discontinuous first-derivative at zero. Therefore, we use a pseudo-absolute function which is defined as follows:

$$
PseudoAbsolute(x) = \begin{cases} 
| x | &: x \geq c \text{ and } x \leq -c \\
\frac{1}{2} c + \frac{1}{(2 \cdot c) \cdot x^2} &: -c < x < c 
\end{cases}
$$

where $c$ is a user-specified, threshold value.

This function has the property that it equals the absolute function's value and first-derivative at $+c$ and $-c$. In addition, the minimum of this function is also the minimum of the absolute function. This turns out to be an effective tool in dealing with the smoothness expectations of the
nonlinear programming code.

5) Gradients

An important consideration must be given to the derivatives as viewed by the nonlinear programming code. Many realistic trajectory problems have derivatives that are difficult to determine analytically. An alternative is to determine those derivatives by finite-differences. However, special care must be taken to ensure that the numerical derivatives are accurate otherwise convergence of the nonlinear optimization problem may not be attained. We initially tried using forward finite-difference algorithms to calculate the first-derivatives; however, this sometimes proved to be problematic for converging to a solution. When we switched to using central finite-differences, the occasional problems were alleviated at the expense of doubling the computational effort.

III. Applications to Formation Initialization and Deployment

A. Model and Cost

The cases discussed in this paper all share a common model of the dynamics. Each spacecraft's motion around the Earth will be affected by a 5x5 spherical-harmonic gravity model using the coefficients from the Joint NASA GSFC and NIMA Geopotential Model, EGM96. The center of mass of the planet will be modeled as the center of the inertial frame. The spacecraft may apply a propulsive force that is maintained constant in magnitude and direction relative to the spacecraft reference frame. The propulsive force will be allowed to change in magnitude and direction at the start of each time segment. The use of the propulsion system will consume propellant and cause the mass of the spacecraft to decrease. The equations of motion between each node are:
\[ \ddot{r} = \sum_{n,m} \ddot{r}_{nm} + \frac{\vec{F}}{m} \]  
\[ \dot{m} = -\|\vec{F}\|/(V_e) \]

where \( \ddot{r} \) is the total acceleration of the position-vector, \( \ddot{r}_{nm} \) is the individual acceleration of position-vector caused by the gravitational attraction of the zonal, sectorial, and tesseral approximations of the Earth's mass\(^4\), \( M_{\text{Earth}} \) is the mass of the Earth, \( \vec{F} \) is the propulsive force vector, \( m \) is the spacecraft's mass, \( \dot{m} \) is the rate of change of the spacecraft's mass, and \( V_e \) is the exit velocity of the propellant. The propulsive force will be constrained to not exceed 5 Newtons.

**Fig. 2** Piecewise-constant-thrusting model. The local-frame thrust is maintained piecewise constant while the spacecraft travels between each node.

For the piecewise-constant thrusting model the cost to be minimized will be the propellant mass consumed by the constant-thrusting. This can be expressed by an integral of propellant mass rate integrated over the course of the trajectory.

\[ \text{Cost} = \int_{t_i}^{t_f} \dot{m} \, dt \]
Since the thrust is maintained constant between nodes, the mass rate will also remain constant between nodes. This leads to a simpler version of the cost equation:

\[
Cost = \sum_{n=1}^{n-1} \int m \, dt = \sum_{n=1}^{n-1} \left| \frac{|Thrust_{node}| f(V_e)}{\Delta t_{node}} \right|
\]  

(10b)

We initially validated our optimization software against solutions available for single-spacecraft scenarios using impulsive-velocity thrusting such as Hohmann and Lambert types of orbital transfers. Our testing showed that our software matched the solutions for both types of orbital transfers.

**B. Spacecraft Formation Cases**

The following cases illustrate how direct multiple-shooting with nonlinear programming can be used to solve cases related to satellite formation design and deployment. Their optimal solutions cannot be solved by classical techniques because the mass of the satellites vary with time and because the nonlinear effects of gravity do not allow linearization at the relative distances that we used.

The first case involves designing a spacecraft formation to have a controlled periodic motion. The subsequent cases compare the effectiveness of deploying the spacecraft into such formations from different initial conditions.
1) Radially-Aligned, Periodic Formation

![Diagram showing radially-aligned, periodic formation]

Fig. 3  Radially-aligned, periodic formation

The motivation for this type of formation stems from such scientific missions as NASA's Space Technology 5 mission and its Magnetospheric Multiscale (MMS) mission. Those missions require the set of three or four satellites to attain a spatial formation either periodically or constantly. The scenario in this section mimics those missions in requiring three spacecraft to periodically align themselves along the radial vector. At the time of formation, the center spacecraft will be required to be at 1.06 Earth Radii (ER), the second spacecraft will be 10 km above it, and the third spacecraft will be 10 km below it. This radial formation will be repeat at periodic intervals along the equatorial plane (i.e. orbital inclination of zero degrees); the
spacecraft are unconstrained during the interval. The periodic interval chosen is equivalent to \( \frac{1}{4} \) the circular orbital period of the center reference orbit (CP). The dynamics of each spacecraft will be modeled by the piecewise-constant-thrusting model previously discussed. All three spacecraft are free to apply a piecewise-constant thrust to satisfy the initial and final alignment conditions. (See Fig. 3)

    Each spacecraft's maximum thrust is constrained to be less than 5 N, and the specific impulse of the propellant is modeled to be 220 s\(^47\). The initial mass of each spacecraft is 100 kg.

_Solution Approach_

    This scenario can be simplified by taking advantage of the periodic nature of the problem. The formation is constrained to repeat its relative positioning at every periodic interval. However, there is no explicit restriction on the velocity of the spacecraft at the beginning of the interval [departure velocity] nor on the velocity of the spacecraft at the end of the interval [arrival velocity]. As such, the optimal trajectory between the two nodes over one periodic interval may be as shown in Fig. 4. If this motion is to be repeated, we desire that the local arrival velocity equal the local departure velocity at the beginning of the next interval. Therefore, we approach this problem by seeking trajectories such that the arrival velocity matches the departure velocity as measured in each spacecraft's local-vertical-local-horizontal frame. By constraining those two velocities to be equal, the controlled trajectory found over one interval will repeat for all subsequent intervals except for deviations caused by the asymmetric distribution of the Earth's mass. Figs. 4 and 5 graphically depict the symmetry conditions that we are seeking.
Fig. 4  Asymmetric trajectory with different departure and arrival velocities as measured in the local reference frames

Fig. 5  Symmetric trajectory with identical departure and arrival velocities as measured in the local reference frames

As mentioned earlier, the formation must attain the radial formation along the equatorial plane. The optimization for the first repeating interval will initially place the formation at 0 longitude and will assume that the orbit is posigrade.

Result

Figure 6 depicts the optimal trajectories for a scenario with a 5-node discretization. The center spacecraft cannot follow an uncontrolled trajectory because the asymmetric gravitational forces cause a deviation from a perfectly circular orbit. The top spacecraft, having to travel a long distance due to its starting and ending along a larger radius, dips lower to travel a shorter distance in the given amount of time; the lower spacecraft follows the opposite behavior under similar reasoning. In comparison to the top spacecraft, the bottom spacecraft requires greater propulsive
thrusting to counteract its higher velocity relative to the center spacecraft, which results from traveling at a lower orbital altitude. As a result, the bottom spacecraft consumes more propellant than the top spacecraft. Therefore, for this radial formation scenario, the primary parameters that affect the propellant consumption, and thereby, the mission life of the formation are: the formation separation distance, the periodic interval, the specific impulse of the propulsion system, and the orbital radius of the center spacecraft.

Fig. 6 Radially-aligned, periodic formation for ¼ CP. Trajectories are shown relative to a translating, rotating frame in a 1.06 ER perfectly circular, reference orbit (shown in gray).
Fig. 7 Thrusting profile of the top, middle, and bottom spacecraft during the first period of
the radial formation at 1.06 ER. Each spacecraft can exert a cross-track, vertical, and horizontal force that is maintained constant relative to a local-vertical local-horizontal reference frame until the next node. The cross-track force is perpendicular to the horizontal and vertical directions.

Table 1. Propellant consumption for the radial formation during the first periodic interval.

<table>
<thead>
<tr>
<th>Propellant Consumption [kg]</th>
<th>Top Spacecraft</th>
<th>Center Spacecraft</th>
<th>Bottom Spacecraft</th>
<th>All spacecraft</th>
</tr>
</thead>
<tbody>
<tr>
<td>Repeating Formation</td>
<td>0.017</td>
<td>0.008</td>
<td>0.032</td>
<td>0.057</td>
</tr>
</tbody>
</table>

2) Formation Deployment Cases

These cases address some options for deploying the aforementioned radial formation. Specifically, we will assume that each of the spacecraft deploy themselves under the influence of the same thrusters used for maintaining the radial formation. The final conditions for all three spacecraft will be the relative positions and velocities determined as in the prior scenario for a 5-node trajectory with a period of ¼ CP.

2.1) Uncoupled Spacecraft Deployment

These scenarios illustrate how the three spacecraft could be deployed independently from each other from various altitudes. This type of scenario could arise if the spacecraft were parked in an orbit while not in radial formation, or if they were aboard a deployment vehicle capable of multiple dispersals of payloads. The scenarios will investigate uncoupled deployment from four radii: 1.06 ER, 10 km above 1.06 ER, 10 km below 1.06 ER, and 1.03 ER. At each of these altitudes, the initial position along these radii is unspecified and is free to be varied by the optimization along with the control history. The control history for each spacecraft is
parameterized into four segments. The initial velocity of the spacecraft, however, are constrained to be the appropriate circular velocity for that altitude. Finally, the deployment duration is constrained to be \( \frac{1}{4} CP \), which is equivalent to 23.05 minutes.

**Results**

The results of the uncoupled deployment into radial formation from various altitudes are shown in Table 2. The lowest propellant consumption for all three spacecraft occurs with uncoupled deployment from 10 km above 1.06 ER. At any given altitude, the optimal deployment location along the initial radius is different for each spacecraft thus resulting in an initial separation of the spacecraft along each orbital radii. This separation is shown in Fig. 8.

**Table 2. Propellant consumption of each spacecraft as it deploys into the radial formation from various deployment radii during \( \frac{1}{4} CP \).** The location along the various radii for releasing each spacecraft from a deployment vehicle is found in the optimization to minimize the propellant consumed by each individual spacecraft.

<table>
<thead>
<tr>
<th>Initial Radius</th>
<th>Initial Velocity</th>
<th>Propellant Consumption [kg]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Top Spacecraft</td>
</tr>
<tr>
<td>1.06 ER + 10 km</td>
<td>Circular velocity</td>
<td>0.014</td>
</tr>
<tr>
<td>1.06 ER</td>
<td>Circular velocity</td>
<td>0.028</td>
</tr>
<tr>
<td>1.06 ER – 10 km</td>
<td>Circular velocity</td>
<td>0.041</td>
</tr>
<tr>
<td>1.03 ER</td>
<td>Circular velocity</td>
<td>0.283</td>
</tr>
</tbody>
</table>
Fig. 8 Left: Positions of the three spacecraft during the ¼ CP deployment from various initial deployment positions to the final radial formation.

Right: Zoomed view of the initial deployment positions. Shown are the optimal deployment positions for each of the spacecraft from the four radii of interest (shown in gray). The optimal deployment positions at a given radius are designated with a different label: square (1.03 ER), circle (10 km below 1.06 ER), triangle (1.06 ER), and diamond (10 km above 1.06 ER). For each of the deployment radii, the optimal deployment location for the top, center, and bottom spacecraft is respectively shown in red, green, and blue. Since none of the trajectories deviate more than 500 meters from the equatorial plane, the positions shown have been projected onto that plane.

For comparison with the next section's scenarios, some further scenarios were investigated. For each of the initial four radii from which the vehicles can be deployed, an optimal deployment location was found for each of the three spacecraft; this results in 12 initial deployment positions (see Fig. 8). For each of these locations, the optimal deployment of the other two spacecraft from the same location is calculated. This has the effect of calculating the cost of a simultaneous, collocated deployment into radial formation from the 12 locations.
Table 3. Propellant consumption of each spacecraft as it deploys into the radial formation from various deployment positions over the course of $\frac{1}{4}$ CP. At each radii, there are three optimal deployment locations that respectively minimize the propellant required to deploy the top, middle, and bottom spacecraft (see Fig. 8). From each of these 12 deployment positions, the other two spacecraft in the final formation are simultaneously released and optimal trajectories into the final radial-formation is sought. This results in 12 scenarios, and the results of each of these scenarios is contained in a row. Note that only deployments from the optimal locations of the center spacecraft are feasible; deployments from the other locations require a violation of the bounded thrusting capabilities for at least one spacecraft.

<table>
<thead>
<tr>
<th>Initial Radius</th>
<th>Initial Velocity</th>
<th>Propellant Consumption [kg]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Top Spacecraft</td>
</tr>
<tr>
<td>1.06 ER + 10 km</td>
<td>Circular velocity</td>
<td></td>
</tr>
<tr>
<td>From optimal deployment location of top spacecraft</td>
<td>0.014</td>
<td>1.722</td>
</tr>
<tr>
<td>From optimal deployment location of center spacecraft</td>
<td>1.783</td>
<td>0.048</td>
</tr>
<tr>
<td>From optimal deployment location of bottom spacecraft</td>
<td>Infeasible</td>
<td>1.804</td>
</tr>
<tr>
<td>1.06 ER</td>
<td>Circular velocity</td>
<td></td>
</tr>
<tr>
<td>From optimal deployment location of top spacecraft</td>
<td>0.028</td>
<td>1.713</td>
</tr>
<tr>
<td>Initial Radius</td>
<td>Initial Velocity</td>
<td>Propellant Consumption [kg]</td>
</tr>
<tr>
<td>----------------</td>
<td>------------------</td>
<td>-----------------------------</td>
</tr>
<tr>
<td>From optimal deployment location of</td>
<td></td>
<td></td>
</tr>
<tr>
<td>center spacecraft</td>
<td>1.795</td>
<td>0.057</td>
</tr>
<tr>
<td>From optimal deployment location of</td>
<td></td>
<td></td>
</tr>
<tr>
<td>bottom spacecraft</td>
<td>Infeasible</td>
<td>1.828</td>
</tr>
<tr>
<td>From optimal deployment location of</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.06 ER – 10 km</td>
<td>Circular velocity</td>
<td>Infeasible</td>
</tr>
<tr>
<td>top spacecraft</td>
<td>0.041</td>
<td>1.704</td>
</tr>
<tr>
<td>From optimal deployment location of</td>
<td></td>
<td></td>
</tr>
<tr>
<td>center spacecraft</td>
<td>1.808</td>
<td>0.066</td>
</tr>
<tr>
<td>From optimal deployment location of</td>
<td></td>
<td></td>
</tr>
<tr>
<td>bottom spacecraft</td>
<td>Infeasible</td>
<td>1.828</td>
</tr>
<tr>
<td>From optimal deployment location of</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.03 ER</td>
<td>Circular velocity</td>
<td>Infeasible</td>
</tr>
<tr>
<td>top spacecraft</td>
<td>0.283</td>
<td>1.589</td>
</tr>
<tr>
<td>From optimal deployment location of</td>
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<td></td>
</tr>
<tr>
<td>center spacecraft</td>
<td>1.819</td>
<td>0.270</td>
</tr>
<tr>
<td>From optimal deployment location of</td>
<td></td>
<td></td>
</tr>
<tr>
<td>bottom spacecraft</td>
<td>Infeasible</td>
<td>1.844</td>
</tr>
</tbody>
</table>

The results shown in Table 3 indicate that some deployments are infeasible. (After the unconstrained optimization, the infeasibility is detected by observing the inequality violations of the thrust levels, and by lack of convergence between the exterior penalty function and the original objective function.) The only collocated deployment location that is feasible is the optimal location for the center spacecraft (see Fig. 9). From this location, the top and lower
spacecraft have sufficient thrusting capabilities to reach the desired end conditions within the
time allotted for deployment. Deploying from the optimal locations for the top spacecraft
requires a thrusting capability for the bottom spacecraft that exceeds 5 N. A similar observation
holds true for the bottom spacecraft.

(a)
Fig. 9  
(a) Deployment of all three spacecraft into the radial formation from the optimal deployment location for the center spacecraft at 10 km above 1.06 ER with circular initial velocity. The trajectories are shown relative to a translating, rotating frame in a 1.06 ER circular, reference orbit (shown in gray). The bands represent the applied propulsive force between each node.

(b) Deployment of all three spacecraft into the radial formation from the optimal deployment location for the center spacecraft at 1.06 ER with circular initial velocity. Format of figure is the same as Part (a).

(c) Deployment of all three spacecraft into the radial formation from the optimal deployment location for the center spacecraft at 10 km below 1.06 ER with circular initial velocity. Format of figure is the same as Part (a).

(d) Deployment of all three spacecraft into the radial formation from the optimal

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**Plot Description**

- **Fig. 9 (a)**: Depicts the deployment of three spacecraft into a radial formation from an optimal deployment location 10 km above 1.06 ER. The trajectories are shown relative to a translating, rotating frame in a 1.06 ER circular reference orbit (gray). The bands indicate the applied propulsive force between nodes.

- **Fig. 9 (b)**: Similar to (a) but for a deployment location 1.06 ER above the spacecraft. Format remains consistent with (a).

- **Fig. 9 (c)**: Illustrates the deployment 10 km below the spacecraft, maintaining the same format as (a).

- **Fig. 9 (d)**: Continues the pattern of deploying spacecraft into radial formation from the optimal location, irrespective of altitude in the diagram.
deployment location for the center spacecraft at 1.03 ER with circular initial velocity. Format of figure is the same as Part (a) with the following exception: the lower trajectory depicts the uncontrolled trajectory emanating from the initial position of the three spacecraft but starting at a circular velocity and subject to the same 5x5 gravity field affecting the three spacecraft.

2.2) Simultaneous Coupled Deployment

Fig. 10 Simultaneous coupled deployment from middle altitude with unconstrained initial velocity

These scenarios illustrate how the three-spacecraft, radial-formation could be simultaneously
deployed from various altitudes. This type of scenario would arise if the spacecraft were aboard a deployment vehicle that was restricted to simultaneously dispersing all of the vehicles. In so doing, the initial velocity of the spacecraft would be very close to the velocity of the deployment vehicle at the time of release. These scenarios will investigate this coupled deployment from four radii: 1.06 ER, 10 km above 1.06 ER, 10 km below 1.06 ER, and 1.03 ER. At each of these altitudes, the initial position along these radii is unspecified and is free to be varied by the optimization along with the control history. The control history for each spacecraft is parameterized into four segments. In half of the scenarios, the initial velocity of the deployment vehicle (which is shared by all three of the spacecraft) is constrained to be a posigrade, circular velocity corresponding to the deployment location. In the other half of the scenarios, the initial velocity is unconstrained in magnitude and direction and is, therefore, an additional parameter in the optimization process.

**Table 4. Scenarios for the coupled deployment of the radial-formation**

<table>
<thead>
<tr>
<th>Initial Radius</th>
<th>Initial Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.06 ER + 10 km</td>
<td>Circular velocity</td>
</tr>
<tr>
<td>1.06 ER + 10 km</td>
<td>Unconstrained velocity</td>
</tr>
<tr>
<td>1.06 ER</td>
<td>Circular velocity</td>
</tr>
<tr>
<td>1.06 ER</td>
<td>Unconstrained velocity</td>
</tr>
<tr>
<td>1.06 ER – 10 km</td>
<td>Circular velocity</td>
</tr>
<tr>
<td>1.06 ER – 10 km</td>
<td>Unconstrained velocity</td>
</tr>
<tr>
<td>1.03 ER</td>
<td>Circular velocity</td>
</tr>
<tr>
<td>1.03 ER</td>
<td>Unconstrained velocity</td>
</tr>
</tbody>
</table>

At the end of the deployment, each spacecraft is to attain the initial conditions appropriate for the radial formation with a 5-node, ¼ CP periodic interval. The deployment duration is also ¼ CP, and is discretized into 5-nodes. Figure 10 depicts the deployment from the middle altitude, 1.06 ER, with an unconstrained velocity.
The goal is to find the optimal deployment position and velocity along the various radii to minimize total propellant consumption.

Results

Table 5 displays the results for the various results. For any particular altitude, the scenario with all vehicles having the same starting unconstrained velocity requires less propellant than the scenario where all vehicles are constrained to start at circular velocity. The additional degree of freedom appears to minimize the total propellant consumption. In all cases, the center spacecraft requires the least amount of propellant to deploy. In fact, so little fuel is required in the unconstrained velocity cases that the center spacecraft trajectory's almost satisfies the end condition position and velocity without any control at all.
Table 5. Propellant consumption of each spacecraft as it deploys into the radial formation from various deployment positions over the course of $\frac{1}{4}$ CP. At each radii, the optimal deployment location is sought that minimizes the combined propellant consumption of all three spacecraft. At this deployment location, all three spacecraft share the same velocity as the deployment vehicle on which they are aboard. At each radii, two cases are inspected: one where the deployment vehicle's initial velocity is constrained to be a posigrade, circular velocity corresponding to the deployment location, and one where the initial velocity is unconstrained and is therefore an additional parameter in the optimization process. In total, there are eight scenarios consisting of four initial deployment radii with two initial deployment velocities.

<table>
<thead>
<tr>
<th>Initial Radius</th>
<th>Initial Velocity</th>
<th>Propellant Consumption [kg]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Top Spacecraft</td>
</tr>
<tr>
<td>1.06 ER + 10 km</td>
<td>Circular velocity</td>
<td>1.783</td>
</tr>
<tr>
<td>1.06 ER + 10 km</td>
<td>Unconstrained velocity</td>
<td>1.757</td>
</tr>
<tr>
<td>1.06 ER</td>
<td>Circular velocity</td>
<td>1.795</td>
</tr>
<tr>
<td>1.06 ER</td>
<td>Unconstrained velocity</td>
<td>1.758</td>
</tr>
<tr>
<td>1.06 ER – 10 km</td>
<td>Circular velocity</td>
<td>1.807</td>
</tr>
<tr>
<td>1.06 ER – 10 km</td>
<td>Unconstrained velocity</td>
<td>1.757</td>
</tr>
<tr>
<td>1.03 ER</td>
<td>Circular velocity</td>
<td>1.815</td>
</tr>
<tr>
<td>1.03 ER</td>
<td>Unconstrained velocity</td>
<td>1.766</td>
</tr>
</tbody>
</table>
A comparison of the results can be made between the coupled deployment from the optimal deployment locations for individual spacecraft (Table 3) versus the coupled deployment from the optimal deployment locations for all three spacecraft (Table 5). Specifically, the four feasible scenarios in Table 3 can be compared with the four scenarios in Table 5 where the initial velocity is constrained to be circular. In the case of the feasible results from Table 3, the deployment location was sought that minimized the propellant consumption of only the center vehicle. In the case of the circular velocity results from Table 5, the deployment location was sought that minimized the propellant consumption of all three vehicles. The results are nearly identical; the largest difference arises in the deployment from 1.03 ER where the coupled deployment of all three spacecraft gains a 0.001 kg improvement over the uncoupled, individual deployment.

A comparison of the results from uncoupled, individual deployment (Table 2) versus coupled deployment (Table 5) shows that uncoupled, individual deployment consumes much less propellant than coupled deployment. This reflects the sensitivity of the propellant consumption for deployment with respect to the difference in total energy between the initial and final states, the deployment time allotted for this transition, and the geometry between the initial and final positions. For the specified deployment time of \( \frac{1}{4} \) CP, there is an optimal position along each deployment radius that minimizes the propellant consumption for each individual spacecraft; these positions were found in Section 2.1. However, the optimal position for coupled deployment (see Fig. 11) is different and this change in position must be compensated for with higher propulsive control.
Fig. 11  Same as Fig. 8 except that the optimal deployment positions for coupled deployment have been added. The optimal deployment location for coupled deployment with unconstrained velocity is shown in magenta. The optimal deployment location for coupled deployment with circular velocity is not shown since it is within tens of meters of the optimal deployment location for the center spacecraft (shown in green).
Fig. 12  (a) Coupled deployment from 10 km above 1.06 ER with unconstrained initial velocity into radial formation. The trajectories are shown relative to a translating, rotating frame in a 1.06 ER circular, reference orbit (shown in gray). The bands represent the applied propulsive force between each node.

(b) Coupled deployment from 1.06 ER with unconstrained initial velocity into radial formation. Format of figure is the same as Part (a).

(c) Coupled deployment from 10 km above 1.06 ER with unconstrained initial velocity into radial formation. Format of figure is the same as Part (a).

(d) Coupled deployment from 1.03 ER with unconstrained initial velocity into radial formation. Format of figure is the same as Part (a) with the following exception: the lower trajectory depicts the uncontrolled trajectory emanating from the initial position of the three spacecraft but starting at a circular velocity and subject to the same 5x5 gravity field affecting the three spacecraft.
Fig. 13  Thrusting profile of the top, middle, and bottom spacecraft during deployment
into radial formation from 1.06 ER with unconstrained initial velocity. Each spacecraft can exert a cross-track, vertical, and horizontal force that is maintained constant relative to a local-vertical local-horizontal reference frame until the next node. The cross-track force is perpendicular to the horizontal and vertical directions. Note that the center spacecraft barely exerts any propulsive force.
Fig. 14 Thrusting profile of the top, middle, and bottom spacecraft during deployment
into radial formation from 1.03 ER with unconstrained initial velocity. Each spacecraft can exert a cross-track, vertical, and horizontal force that is maintained constant relative to a local-vertical local-horizontal reference frame until the next node. The cross-track force is perpendicular to the horizontal and vertical directions. Note that the center spacecraft barely exerts any propulsive force.

C. Summary

Deployment

The deployment of a radial-formation from four initial radii with different initial velocities was considered. The uncoupled, $\frac{1}{4}$ CP duration, optimal deployment for each spacecraft from a circular parking orbit was investigated for four radii. At each of these radii, the optimal deployment position along each radii was sought to minimize the propellant consumption of any particular spacecraft to reach the position and velocity needed to initiate the radial formation. In so doing, no consideration was made for the combined, three-vehicle propellant consumption. Figure 8 shows the optimal deployment locations for each individual spacecraft at the four radii investigated. Table 2 shows the propellant consumed by each spacecraft at each of those locations. The results indicate that deployment from 10 km above 1.06 ER was the most propellant-optimal, circular orbit from which to independently deploy all three spacecraft.

The coupled, $\frac{1}{4}$ CP duration, optimal deployment of all three vehicles was investigated for the same four radii. At each of these radii, the optimal deployment position along each radii was sought to minimize the propellant consumption of all three spacecraft such that the spacecraft could reach the position and velocity needed to initiate the radial formation. Furthermore, for half of the scenarios the optimal deployment velocity was also sought to further minimize the combined propellant consumption. It was found (see Table 5) that coupled deployment from 1.03
ER with an unconstrained initial velocity yielded the lowest propellant consumption for the deployment of all three vehicles.

*Initial Guesses*

As mentioned in Sec. II.B.1, using direct multiple-shooting with nonlinear programming, or using a similar direct approach to solve optimal control problems, requires that the nonlinear programming code be provided with a guess for the solution. In this paper, the initial guess for the control history was always the zero-thrust control profile; the initial guess for the position was along the constrained initial radius with a true anomaly 90 degrees earlier than the final true anomaly; the initial guess for velocity was the circular velocity at the initial position. We have found that our optimization method provided good optimal solutions. There is no assurance, however, that local solutions found from this initial guess will lead to the global solution; hence, our zero-control guesses may may bias the solutions found. Therefore, another approach to providing guesses could involve the use of genetic algorithms\textsuperscript{48} The algorithms could be used on their own to determine near-optimal trajectories\textsuperscript{49}, or they could be used to find guesses that are then refined by another numerical technique, such as the one used in this paper, to find globally-optimal trajectories.\textsuperscript{50}
**Computational Cost**

The duration of numerically solving each scenario depended on the number of spacecraft involved, the number of nodes for each control history, and the duration of the scenario. Increasing the number of spacecraft, each with a discretized control history, obviously causes the need for greater computation time. Similarly, increasing the number of nodes creates more optimization variables which increases the optimization duration. Increasing the duration of the scenario required more computational effort because it forced the underlying numerical integrator to use more time-steps to propagate the system states.

The exterior penalty function, which converts a constrained optimization problem into an unconstrained optimization problem, requires that a sequence of unconstrained problems be solved with an ever-increasing penalty parameter so that the unconstrained problem approaches the solution to the constrained optimization problem. Decreasing the desired errors in position and velocity thus requires increasing the number of unconstrained optimizations. For all of the scenarios shown in the paper, three to four unconstrained optimizations were used to attain position errors less than 5 cm and velocity errors less than 3 mm/s at the end time.

Most all of the computations were run on a x86-compatible computer using an AMD XP 2500+ processor. Computation times ranged from 1 minute for a single-spacecraft, 5-node, ¼ CP scenario to 28 minutes for a three-spacecraft, 5-node, ¼ CP scenario.

**Formation Initialization**

The radial formation investigated requires 0.057 kg for the first ¼ CP period. The primary parameters affecting the amount of propellant required to maintain this periodic formation are: the formation separation distance, the periodic interval, the specific impulse of the propulsion system, and the orbital radius of the center vehicle. If the formation need only be maintained
during certain times or events, then the formation could assume an uncontrolled trajectory in a parking orbit, as mentioned in Ref. 28. Then, when the radial formation needs to be assembled, each spacecraft could individually deploy as investigated in Section III.B.2.2.

**Optimal Deployment Positions**

As discussed in results for Section III.B.2.2, at each of the four circular orbits inspected, the optimal deployment location for the center spacecraft was only tens of meters away from the optimal deployment location for all three spacecraft of the radial formation. Relatedly, Section III.B.2.1 discusses the attempt to deploy all of the spacecraft from the optimal deployment location for the top or bottom spacecraft; this was infeasible due to the inability of the bottom or top spacecraft, respectively, to reach its intended position and velocity for the radial formation within ¼ CP given the bounded, thrusting capability of each spacecraft.

**Commonality of Relative Trajectories**

The trajectories of the top and bottom spacecraft relative to the center spacecraft from the eight coupled deployment positions (enumerated in Table 5) are shown in Fig. 15. Note that the transverse dynamics have similar characteristics in all of the scenarios.
Fig. 15  The relative trajectories of the top and bottom spacecraft relative to the center spacecraft. The trajectories from all eight of the coupled deployments are shown here. Since none of the trajectories deviate more than 500 meters from the equatorial plane, these trajectories have been projected onto that plane.

Conclusion

The initialization and deployment of a radial formation was investigated using direct multiple-shooting with nonlinear programming. This proved to be an effective tool because of the relative ease with which different constraints and coupled conditions could be introduced. The uncoupled deployment of individual spacecraft over the course of ¼ CP required less propellant than the coupled deployment of all three spacecraft. Among the coupled deployment scenarios, the scenarios where the deployment vehicle's velocity was unconstrained yielded a lower propellant consumption than when it was constrained to circular velocity. Under all of the coupled
deployment scenarios investigated, the relative trajectories of the top and bottom spacecraft with respect to the center spacecraft were very similar. This paper has shown how direct multiple-shooting with nonlinear programming can be applied to find formations with controlled, periodic behavior using piecewise-constant thrusting. The paper has also demonstrated the algorithm's application to testing various optimal deployment strategies with varying conditions and simultaneous constraints that cannot be solved analytically using linearized methods.
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